Optimal Consumption under Deterministic Income

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Abstract

We consider an individual or household endowed with an initial wealth, having an income and consuming goods and services. The wealth development rate is assumed to be a deterministic continuous function of time. The objective is to maximize the discounted consumption. Via the Hamilton–Jacobi–Bellman approach we prove the existence and the uniqueness of the solution to the considered problem in the viscosity sense. Furthermore we derive an algorithm for explicit calculation of the value function and optimal strategy. It turns out that the value function is in general not continuous. The method is illustrated by two examples.

Key words: optimal control, optimal consumption, value function, Hamilton–Jacobi–Bellman equation, viscosity solutions, semi-continuous envelopes.

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1 Introduction

Maximizing the expected utility of an individual from consumption and by controlling investment has been a classical problem in mathematical finance for a long time. The interested reader is referred to papers by Karatzas et al. [8, 9] or Cox and Huang [5].

In actuarial science consumption is often interpreted as dividend payout. Numerous papers and books have been written on the topic of dividend maximization in the framework of the classical risk model, its diffusion approximation or piecewise deterministic Markov processes. A summarization of actuarial findings of the last 50 years can be found in Avanzi [1] or Albrecher and Thonhauser [2].

In this paper we consider an individual or household whose income stream is described by a deterministic process with continuous drift function. The drift function can attain negative

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values or be even periodic, whereas the assumption of a non-negative drift is common in literature. A suitable example provide households with income depending on seasonal agriculture or tourism, which is characteristic for developing countries.

We assume that the primal interest of the individual/household is to maximize the cumulated value of discounted consumption from a given time up to a finite time horizon. Or, in other words, to maximize cumulated discounted utility from consumption, given a linear utility function. One may notice that in the literature the dividend maximization problem is usually stated on an infinite time horizon. The present problem formulation can be regarded as a non stochastic limiting case of Grandits [7], who deals with a pure consumption maximization problem on a finite time horizon for a diffusion type wealth process.

Upon first sight the problem seems to be relatively easy to solve. However, some difficulties arise such as that the value function turns out to be discontinuous even in semi-continuity sense. Furthermore for applying the viscosity solution approach to the associated Hamilton–Jacobi–Bellman equation we also have to take into account the discontinuity of the considered value function. For semi-continuous viscosity solutions the problem of existence and uniqueness of a solution to Hamilton–Jacobi–Bellman equations with convex Hamiltonians was dealt with by Barron and Jensen [3]. There the main idea is to transfer the uniqueness requirement on solutions to their lower semi-continuous envelopes. In this paper we will first use the concept of weak comparison, described for example in Fleming and Soner [6], and finally show the strong uniqueness using specific properties of the value function. For a general introduction into the theory of viscosity solutions see for example Bardi and Capuzzo-Dolcetta [4].

The contribution of the present paper, beyond the discussion of the HJB approach, is to establish an algorithm that allows to determine a closed form expression for the value function and the optimal strategy.

The paper is structured as follows. At first we give a mathematical formulation of the model and state some important properties of the value function. Section 2, which is the main part of the paper, is dedicated to algorithm derivation. The Hamilton–Jacobi–Bellman approach is discussed in Section 3. For the sake of clarity of presentation we postpone the proofs of this section to an Appendix. Two illustrative examples are given in Section 4.

Let us now start with the model formulation. The deterministic wealth process minus consumption is given by:

- $\mathrm{d}X_t^C = \mu_t \,\mathrm{d}t - \,\mathrm{d}C_t$ with $X_{0-} = x$ and for $0 \le t \le T$,
- μ_t is continuous on $[0,T]$ with only finitely many zeros in $[0,T]$,
- $C = (C_t)_{t\in[0,T]}$ is cumulated consumption, càdlàg, increasing, $\Delta C_s \le X_{s-}^C$.

The value of a given strategy is given by

$$J(0,x,C) = \int_{0-}^{\tau-} e^{-\beta t} \,\mathrm{d}C_t + e^{-\beta\tau} X_\tau^C \ ,$$

where $\beta > 0$ is some discounting rate and $\tau = \inf\{t > 0 \,|\, X_t^C < 0\} \wedge T$. Of course τ depends on C, if some distinctions are needed we will indicate them.

For application/derivation of some dynamic programming principle we need

$$\begin{aligned} X_s^C &= x + \int_t^s \mu_r \,\mathrm{d}r - C_s, \qquad \text{for } 0 \le t \le s \le T \quad \text{and } X_{t-} = x \ , \\ J(t,x,C) &= \int_{t-}^{\tau-} e^{-\beta s} \,\mathrm{d}C_s + e^{-\beta\tau} X_\tau^C \ . \end{aligned}$$

We tacitly assume the adaptions on the definitions of τ and C. We write $\mathcal{C}(t,x)$ for the set of admissible consumption strategies when starting at time t at level $x \ge 0$. The value function of the associated maximization problem is given by

$$\begin{aligned} &V(t,x) = \sup_{C\in\mathcal{C}(t,x)} J(t,x,C) \ \text{ for } (t,x) \in [0,T) \times [0,\infty) \ , \\ &V(T,x) = e^{-\beta T} x \ \text{ for } x \in [0,\infty) \ , \qquad (1) \\ &V(t,x) = 0 \ \text{ for } (t,x) \in [0,T] \times (-\infty,0) \ . \end{aligned}$$

In the following we will denote the requirements $V(t,x) = 0$ for $(t,x) \in [0,T] \times (-\infty,0)$ and $V(T,x) = e^{-\beta T} x$ for $x \in [0,\infty)$ by **(P1)**.

For later purpose we mention that for $s \ge \tau$ we have $C_s = C_{\tau-}$ and $X_s = X_\tau$, i.e. consumption stops at the event of ruin.

The reader may notice that we assume a strategy to be càdlàg and hence the controlled process X^C as a post-consumption process, compare Schmidli [11, p. 80]. As a consequence we have to include a possible initial consumption $C_0 > 0$ and to exclude a too large consumption leading to ruin in the value function. In the following Lemma we state useful properties of the value function, which can be obtained immediately from the model assumptions.

Lemma 1.1

The value function $V(t,x)$ fulfils

- $V(t,x)$ *is increasing in* x, **(P2)**
- $V(t,x) \leq e^{-\beta t}x + \lambda(t)$ *for* $\lambda(t) = \int_t^T |\mu_s| e^{-\beta s}\, \mathrm{d}s$. **(P3)**

Proof: Let $y > x$ and C be an ε-optimal strategy at (t,x), i.e. $V(t,x) \leq V^C(t,x) + \varepsilon$. For initial capital y at t construct a strategy $\tilde{C}$ as follows: payout $y - x$ immediately and follow the strategy C. Thus, we have

$$V(t,y) - V(t,x) \geq V^{\tilde{C}}(t,y) - V^C(t,x) - \varepsilon = (y-x)e^{-\beta t} - \varepsilon\ .$$

Because ε was arbitrary, we obtain the result.

For every admissible consumption strategy C it holds

$$\int_{t-}^{\tau-} e^{-\beta s}\, \mathrm{d}C_s + e^{-\beta\tau}X_\tau^C \leq xe^{-\beta t} + \int_t^\tau e^{-\beta s}|\mu_s|\, \mathrm{d}s\,.$$

It follows $V(t,x) \leq e^{-\beta t}x + \int_t^T |\mu_s| e^{-\beta s}\, \mathrm{d}s$. □

2 Optimal Strategy - Construction of a Solution

The following Lemma turns out to be crucial for the construction of the optimal consumption strategy.

Lemma 2.1

Assume that in $(t,x) \in [0,T] \times [0,\infty)$ *it is optimal to payout* ΔC_t. *Then for* (t,y) *with* $y \in (x - \Delta C_t, x)$ *it is optimal to payout* $y - x + \Delta C_t$ *and to continue with the optimal strategy for the point* (t,x).

Proof: We have that

$$V(t, x - \Delta C_t) = V(t,x) - e^{-\beta t}\Delta C_t\ .$$

By the assumption of the Lemma we get for the value of the strategy C^* (which for $y \in (x - \Delta C_t, x)$ pays $y - x + \Delta C_t$) that

$$
\begin{aligned}
J(t, y, C^*) &= V(t, x - \Delta C_t) + e^{-\beta t}(y - x + \Delta C_t) \\
&= V(t, x) + e^{-\beta t}(y - x) \; .
\end{aligned}
$$

Assume that there is some policy $\tilde{C}$ such that

$$
J(t, y, \tilde{C}) > J(t, y, C^*).
$$

Now define a strategy $\hat{C}$ for initial point (t, x) as follows: payout $y - x$ and continue with $\tilde{C}$. We derive:

$$
\begin{aligned}
J(t, x, \hat{C}) &= e^{-\beta t}(x - y) + J(t, y, \tilde{C}) \\
&> e^{-\beta t}(x - y) + J(t, y, C^*) = V(t, x),
\end{aligned}
$$

which yields a contradiction to the optimality of the payment ΔC_t for (t, x). $\square$

<u>Assertion</u>:

The value function $V(t, x)$ is determined by one of the two following cases:

A

$$
V(t, x) = e^{-\beta t}x + \alpha_0 I_{[0,\gamma_1)}(x) + \alpha_1 I_{[\gamma_1,\gamma_2)}(x) + \ldots + \alpha_n I_{[\gamma_n,\infty)}(x) \; ,
$$

where $n \in \mathbb{N}_0$ ($n = 0$ has the consequence $V(t, x) = e^{-\beta t}x + \alpha_0$) and

- $0 \leq \alpha_0 < \alpha_1 < \ldots < \alpha_n$,
- $0 < \gamma_1 < \gamma_2 < \ldots < \gamma_n$,
- $\mu_s \geq 0$ for all $s \in [t - \varepsilon, t]$ for some $\varepsilon > 0$.

B

$$
V(t, x) = e^{-\beta t}x + \alpha_0 I_{[0,\gamma_1)}(x) + \alpha_1 I_{[\gamma_1,\gamma_2)}(x) + \ldots + \alpha_n I_{[\gamma_n,\infty)}(x) \; ,
$$

where again $n \in \mathbb{N}_0$ and

- $0 \leq \alpha_0 < \alpha_1 < \ldots < \alpha_n$,

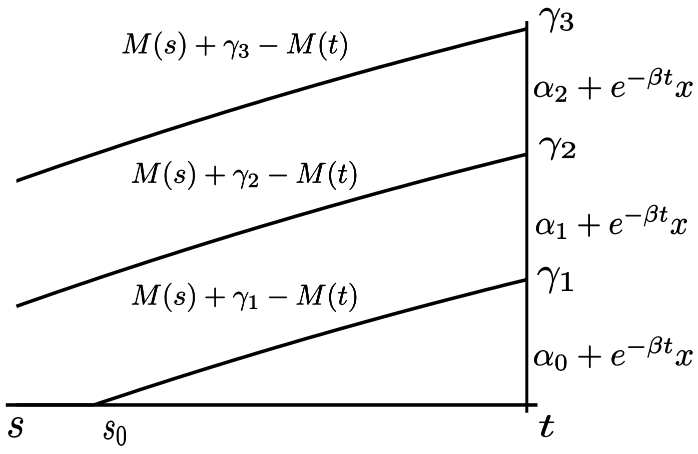

Figure 1: Situation in case A

- $0 < \gamma_1 < \gamma_2 < \ldots < \gamma_n$,
- $\mu_s \leq 0$ for all $s \in [t-\varepsilon, t]$ for some $\varepsilon > 0$.

We are going to prove this assertion by showing that if starting in situation **A** or **B**, $V(t,x)$ again is of that type if time runs backward. In total we derive an algorithm which, starting with $V(T,x)$ (at time T we are either in situation **A** or **B** depending on the sign of μ_T with $n=0$ and $\alpha_0=0$), constructs the whole value function and optimal strategy.

Proof: Assume $\mu(T) > 0$ or $\lim_{t\to T} \mathrm{sgn}(\mu(t)) = 1$, i.e. we are in case **A**.
Let $M(t) = \int_0^t \mu_r \,\mathrm{d}r$ and $\bar{s} = \sup\{s < t \,|\, \mu_s < 0\}$ be the last time before t where the drift changes its sign. Furthermore define

$$
\begin{aligned}
s_0 &= \sup\{s < t \,|\, M(s) + \gamma_1 - M(t) = 0\}\ , \\
s_1 &= \sup\{s < t \,|\, e^{-\beta s}(M(s) + \gamma_1 - M(t)) + \int_s^t e^{-\beta r}\mu_r\, dr + \alpha_0 > e^{-\beta t}\gamma_1 + \alpha_1\} \\
&= \sup\{s < t \,|\, \int_s^t \mu_r(e^{-\beta r} - e^{-\beta s})dr > \gamma_1(e^{-\beta t} - e^{-\beta s}) + \alpha_1 - \alpha_0\}\ .
\end{aligned}
$$

The point in time s_1 is the *first* time on the first curve (given by $M(s)+\gamma_1-M(t)$, going backward in time from t) where it is preferable to payout everything and consume the drift up to time t instead of staying there, reaching the point (t,γ_1) where one receives $e^{-\beta t}\gamma_1 + \alpha_1$. Figure 1 illustrates the specific situation of case **A**.

Let $s^* = \max\{\bar{s}, s_0, s_1\}$. At first we are going to look at the problem on the set:

$$\{(s,x) \,|\, s^* \leq s < t,\, 0 \leq x < M(s) + \gamma_1 - M(t)\}\ ,$$

where we assert that it is optimal to payout everything and to stay on the x-axis (i.e. consume the drift). This strategy C^* is determined by:

$$\begin{cases} \Delta C^*_s = X_{s-}, & \\ \dot{C}^*_r = \mu_r, & r \in [s,t) \ . \end{cases}$$

Assume from time $s < t$ on we follow an arbitrary strategy C and switch to the optimal one at time t. Let $\tau = \inf\{r > s \,|\, X^C_r < 0\} \wedge t$, we have

$$\begin{aligned} J(s,y,C) &= \int_{s-}^{\tau-} e^{-\beta r}\,\mathrm{d}C_r + V(t, X^C_{t-}) I_{\{\tau = t\}} \\ &= \int_{s-}^{\tau-} e^{-\beta r}\,(-\,\mathrm{d}X^C_r + \mu_r\,\mathrm{d}r) + V(t, X^C_{t-}) I_{\{\tau = t\}} \ . \end{aligned}$$

Using $N(s) = \int_0^s e^{-\beta r}\mu_r\,\mathrm{d}r$ and integration by parts we derive

$$\begin{aligned} J(s,y,C) =& N(\tau) - N(s) - \left\{ e^{-\beta r} X^C_r \Big|_{r=s-}^{r=\tau-} + \beta \int_s^\tau e^{-\beta r} X^C_r\,\mathrm{d}r \right\} + V(t, X_{t-}) I_{\{\tau=t\}} \\ =& N(\tau) - N(s) + e^{-\beta s} X^C_{s-} - e^{-\beta\tau} X^C_{\tau-} - \beta \int_s^\tau e^{-\beta r} X^C_r\,\mathrm{d}r \\ &+ I_{\{\tau=t\}} \left(e^{-\beta t} X^C_{t-} + \sum_{i=0}^n \alpha_i I_{[\gamma_i,\gamma_{i+1})}(X^C_{t-}) \right) \ . \end{aligned} \tag{2}$$

Since $s \geq s^* \geq s_0$ the level γ_1 can not be reached by X^C, therefore (2) is equivalent to

$$\begin{aligned} J(s,y,C) &= N(\tau) - N(s) + e^{-\beta s} X^C_{s-} - e^{-\beta\tau} X^C_{\tau-} - \beta \int_s^\tau e^{-\beta r} X^C_r\,\mathrm{d}r \\ &\quad + I_{\{\tau=t\}} \left(e^{-\beta t} X^C_{t-} + \alpha_0 \right) \\ &\leq N(t) - N(s) + e^{-\beta s} X^C_{s-} + \alpha_0 \ . \end{aligned} \tag{3}$$

The last inequality is due to the fact that $N(\cdot)$ is increasing. We observe that there is an equality in (3) for the above defined strategy C^*, which yields that $V(s,x) = e^{-\beta s}x + \alpha_{0,\mathrm{new}}$ with $\alpha_{0,\mathrm{new}} = \alpha_0 + N(t) - N(s)$, i.e. $V(s,x)$ is again of the claimed form.

Now we look at points $\{(s,x) \,|\, s^* \leq s < t,\, x = M(s) + \gamma_1 - M(t)\}$, here the level γ_1 can be reached. Instead of (3) we have

$$N(\tau) - N(s) + e^{-\beta s} X^C_{s-} - \beta \int_s^\tau e^{-\beta r} X^C_r\,\mathrm{d}r + I_{\{\tau=t,\, X^C_{t-} < \gamma_1\}}\alpha_0 + I_{\{\tau=t,\, X^C_{t-} = \gamma_1\}}\alpha_1 \ . \tag{4}$$

Suppose there is some $r \in [s,t]$ with $X^C_r < M(r) + \gamma_1 - M(t)$, then at time t level γ_1 can not be attained and (4) is smaller than

$$N(t) - N(s) + e^{-\beta s} X^C_{s-} + \alpha_0.$$

Whereas the policy staying on the curve, doing nothing, delivers the value $e^{-\beta t}\gamma_1 + \alpha_1$. Since $s \geq s^* \geq s_1$ the last policy yields a higher value such that

$$V(s, M(s) + \gamma_1 - M(t)) = e^{-\beta t}\gamma_1 + \alpha_1, \text{ for } s^* \leq s < t \,.$$

As a first consequence we have that V is not continuous along the curve $M(s)+\gamma_1-M(t)$.

In a second step we deal with points which are in between the first curve $M(s)+\gamma_1-M(t)$ and the second one $M(s)+\gamma_2-M(t)$. As we know from the above discussion it is optimal to stay on the first curve, in combination with Lemma 2.1 we obtain that it is not optimal to jump from above this curve to a level below.

Define

$$s_2 = \sup\{s < t \,|\, (\gamma_2 - \gamma_1)e^{-\beta s} + \alpha_1 + e^{-\beta t}\gamma_1 > \alpha_2 + e^{-\beta t}\gamma_2\}, \tag{5}$$

which is the first time (going backwards from t) such that it is as good to stay on the second curve as to jump down to the first curve and stay there. Actually this curve vanishes in s_2 together with the associated discontinuity of V.

Substituting the level $x = 0$ by the first curve in the previous step of the proof we obtain in an analogue way that for $s \geq s^* \vee s_2$ if $M(s) + \gamma_1 - M(t) < x < M(s) + \gamma_2 - M(t)$ it is optimal to jump down to the first curve and stay there. If x is already on the second curve it is optimal to stay there up to time t.

An application of these thoughts to areas between higher curves $M(s) + \gamma_j - M(t)$ proves the claimed structure of $V(s, x)$ and determines the optimal policy. In total we get with s_k $k = 3, \ldots, n$ defined like s_2 in (5):

- if for some index j we have $s_j > \bar{s}$, then the line of discontinuity given by $M(s) + \gamma_j - M(t)$ vanishes before a switch to case **B**
- if $s_0 > \bar{s}$, then the first line of discontinuity on the time axis vanishes (this may happen as well for the higher curves "later", if these curves still exist.)
- in case $\mu_s \geq 0$ the number of discontinuities can only decrease.

Now we can deal with the assumption that $V(t, x)$ is in case **B**. Let

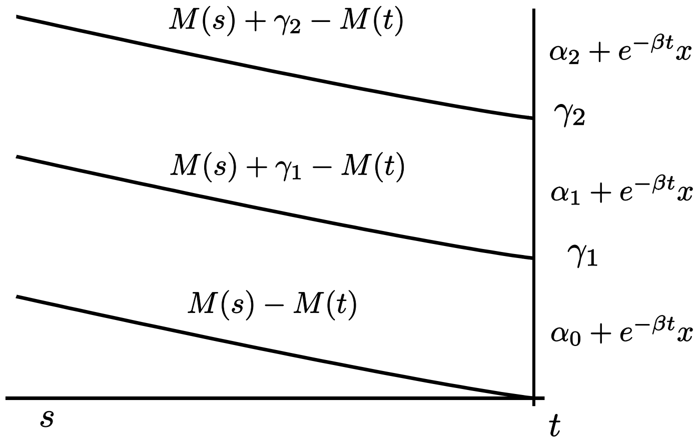

Figure 2: Situation in case B

$$
\begin{aligned}
\bar{s} &= \sup\{s < t \,|\, \mu_s > 0\},\\
s_0 &= \sup\{s < t \,|\, e^{-\beta s}(M(s) - M(t)) > \alpha_0\},\\
s^* &= \max\{s_0, \bar{s}\},
\end{aligned}
$$

notice s_0 is the first point in time from t backwards, where it is better to leave the lowest curve $M(s) - M(t)$ by paying out everything instead of waiting until time t. The situation containing the curves $M(s) + \gamma_j - M(t)$ is illustrated in Figure 2.

We claim that on $\{(s, x) \,|\, s^* \leq s < t,\, 0 \leq x < M(s) - M(t)\}$ it is optimal to payout everything immediately. For an arbitrary strategy C we have as in case **A** that:

$$
\begin{aligned}
J(s, x, C) =& N(\tau) - N(s) + e^{-\beta s} X^C_{s-} - e^{-\beta\tau} X^C_{\tau-} - \beta \int_s^\tau e^{-\beta r} X^C_r \,\mathrm{d}r\\
&+ I_{\{\tau = t\}} \left(e^{-\beta t} X^C_{t-} + \sum_{i=0}^{n} \alpha_i I_{[\gamma_i, \gamma_{i+1})}(X^C_{t-}) \right).
\end{aligned} \tag{6}
$$

Since $s \geq s^*$ and $x < M(s) - M(t)$ ruin happens before time t, therefore (6) is equal to

$$
N(\tau) - N(s) + e^{-\beta s} X^C_{s-} - \beta \int_s^\tau e^{-\beta r} X^C_r \,\mathrm{d}r \leq e^{-\beta s} X^C_{s-}.
$$

The last equality holds since $N(\cdot)$ is decreasing, the "$\leq$" changes to a "$=$" in the case everything is paid out immediately. Therefore $V(s, x) = e^{-\beta s} x$ on $\{(s, x) \,|\, s^* \leq s < t,\, 0 \leq x < M(s) - M(t)\}$.

Now look at $\{(s, x) \,|\, s^* \leq s < t,\, 0 \leq x = M(s) - M(t)\}$ (points on the lowest curve), in this

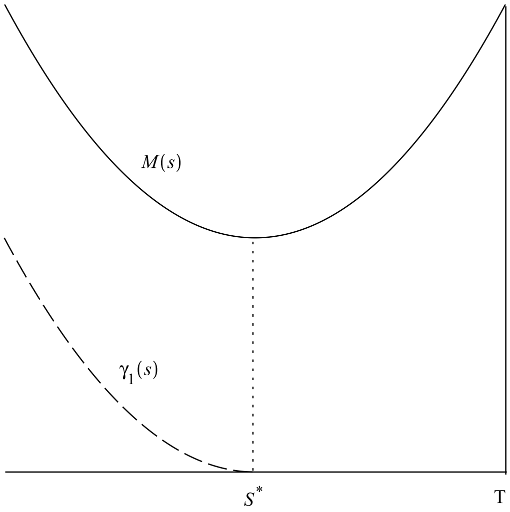

Figure 3: The first discontinuity curve $\gamma_1(s)$.

case $(t,0)$ can be reached. We have

$$J(s,x,C) = N(\tau) - N(s) + e^{-\beta s}X^C_{s-} - e^{-\beta\tau}X^C_{\tau-} - \beta\int_s^\tau e^{-\beta r}X^C_r \,\mathrm{d}r + I_{\{\tau=t\}}\alpha_0. \tag{7}$$

If there is some $r \in [s,t]$ such that $X^C_r < M(r) - M(t)$, one cannot reach $(t,0)$ and (7) is smaller or equal to $e^{-\beta s}X^C_s$. Staying on the curve gives the value α_0. Because of $s \ge s^* \ge s_0$ this yields the higher value and $V(s, M(s) - M(t)) = \alpha_0$ for $s^* \le s < t$. If $\alpha_0 > 0$ a discontinuity along $M(s) - M(t)$ for $s < t$ is generated in $(t,0)$ which vanishes at time s_0.

The areas between the following higher curves can be treated as in case **A**. □

In the remark below we sum up some important properties of the value function following from the above proof.

Remark 2.2

- *If* $\mu(T) > 0$ *or* $\lim_{t\to T} \operatorname{sgn}(\mu(t)) = 1$, *then the value function is continuous on* $(s^*,T] \times [0,\infty)$, *where* $s^* = \sup\{s \in [0,T) : \ \mu_s < 0\}$. *On* $(s^*,T] \times [0,\infty)$ *it is optimal to payout everything and the value function is given by*

$$V(t,x) = e^{-\beta t}x + \int_t^T e^{-\beta s}\mu(s)\,\mathrm{d}s\ .$$

  In particular, the value function is continuous if $\mu(t) \ge 0$ *for all* $t \in [0,T]$. **(P4)**

  Assume $s^* > 0$ *and* $\alpha_0(s^*) = \int_{s^*}^T e^{-\beta s}\mu_s\,\mathrm{d}s > 0$. *Then the first discontinuity curve is* *given by* $\gamma_1(s) = -\int_s^{s^*} \mu_r \, \mathrm{d}r$. *In Figure 3 we see the function* $M(s) = \int_s^T \mu_r \, \mathrm{d}r$ *and the first discontinuity curve* $\gamma_1(s)$ *starting in* s^*.

- *The value function is right continuous in the* x*-component with*

$$\lim_{h\to 0} \frac{V(t,x+h) - V(t,x)}{h} = e^{-\beta t} \, . \qquad \textbf{(P5)}$$

- *There exist* $0 = t_{m+1} < ... < t_1 = T$ *and continuously differentiable, either strictly increasing or strictly decreasing functions* $0 < \gamma_{2,1} < ... < \gamma_{2,n_2}, ..., 0 < \gamma_{m+1,1} < ... < \gamma_{m+1,n_{m+1}}$ *such that* $V(t,x)$ *is continuous on* $[0,T] \times [0,\infty) \backslash S$ *with* $S = \bigcup_{j=2}^{m+1} \bigcup_{i=1}^{n_j} \{(s, \gamma_{j,i}(s)), \ s \in [t_j, t_{j-1})\}$. *Furthermore,* V *is continuously differentiable in* x *on every set* $\{(s,x): \ t_j < s < t_{j-1}, \ \gamma_{j,i-1}(s) < x < \gamma_{j,i}(s)\}$. **(P6)**

3 Dynamic programming - heuristics for Hamilton–Jacobi–Bellman equation

As starting point for the derivation of some Hamilton–Jacobi–Bellman (HJB) equation we need the following dynamic programming principle:

$$V(t,x) = \sup_{C\in\mathcal{C}(t,x)} \left\{ \int_{t-}^{\bar{T}\wedge\tau-} e^{-\beta s} \, \mathrm{d}C_s + V(\bar{T}\wedge\tau, X^C_{\bar{T}\wedge\tau-}) \right\} \, , \qquad (8)$$

for $t \le \bar{T} \le T$.

Proof: Let $C \in \mathcal{C}(t,x)$, then

$$\begin{aligned} J(t,x,C) &= \left(\int_{t-}^{\tau-} e^{-\beta s} \, \mathrm{d}C_s + e^{-\beta\tau} X_\tau \right) I_{\{\tau \le \bar{T}\}} \\ &\quad + \left(\int_{t-}^{\bar{T}-} e^{-\beta s} \, \mathrm{d}C_s + \int_{\bar{T}-}^{\tau-} e^{-\beta s} \, \mathrm{d}C_s + e^{-\beta\tau} X_\tau \right) I_{\{\tau > \bar{T}\}} \\ &= \int_{t-}^{\bar{T}\wedge\tau-} e^{-\beta s} \, \mathrm{d}C_s + J(\bar{T}\wedge\tau, X^C_{\bar{T}\wedge\tau-}, C) \, , \end{aligned}$$

where strategy C is taken for $\bar{T} \le s \le \tau$ (just the C from $\bar{T}$ onwards). Therefore obviously we have:

$$V(t,x) \le \sup_{C\in\mathcal{C}(t,x)} \left\{ \int_{t-}^{\bar{T}\wedge\tau-} e^{-\beta s} \, \mathrm{d}C_s + V(\bar{T}\wedge\tau, X^C_{\bar{T}\wedge\tau-}) \right\} \, .$$

Now set $W(t,x)$ to be equal to the right hand side of (8) and let C^* be an $\varepsilon/2 > 0$ optimal strategy for it,

$$W(t,x) - \frac{\varepsilon}{2} \leq \int_{t-}^{\bar{T}\wedge\tau-} e^{-\beta s}\, \mathrm{d}C^*_s + V(\bar{T}\wedge\tau, X^{C^*}_{\bar{T}\wedge\tau-}).$$

Since everything is deterministic we can choose again an $\varepsilon/2 > 0$ optimal strategy $\bar{C}$ for $(\bar{T}\wedge\tau, X^{C^*}_{\bar{T}\wedge\tau-})$ such that $V(\bar{T}\wedge\tau, X^{C^*}_{\bar{T}\wedge\tau-}) - \varepsilon/2 \leq J(\bar{T}\wedge\tau, X^{C^*}_{\bar{T}\wedge\tau-}, \bar{C})$. Then

$$\begin{aligned} W(t,x) - \varepsilon &\leq \int_{t-}^{\bar{T}\wedge\tau^{C^*}-} e^{-\beta s}\, \mathrm{d}C^*_s + \int_{\bar{T}\wedge\tau^{C^*}-}^{\tau^{\bar{C}}-} e^{-\beta s}\, \mathrm{d}\bar{C}_s + e^{-\beta(\tau^{\bar{C}}\wedge\tau^{C^*})} X^{\tilde{C}}_{\left(\tau^{\bar{C}}\wedge\tau^{C^*}\right)} \\ &= J(t,x,\tilde{C}) \leq V(t,x) \end{aligned}$$

where $X^{\tilde{C}}$ results from taking strategy C^* form t to $\bar{T}$ and if not ruined before going on with $\bar{C}$, i.e. $\tilde{C}_s = C^*_s\, I_{\{t\leq s<\bar{T}\}} + \bar{C}_s\, I_{\{\bar{T}\leq s\leq T\}}$ and stopping it if ruin occurs. Therefore for every $\varepsilon > 0$ we have

$$W(t,x) - \varepsilon \leq V(t,x) \leq W(t,x)\ ,$$

which proves (8). □

Now we can in a heuristic way derive the associated HJB equation. Suppose $V(t,x) \in C^{1,1}([0,T]\times[0,\infty))$ and that $C_s = \int_t^s c_z\, \mathrm{d}z$ for some non-negative and continuous density $c : [t,T] \to \mathbb{R}^+$. Let C be an $\varepsilon > 0$ optimal strategy for $V(t,x)$ $(x > 0)$, then

$$V(t,x) - \varepsilon \leq \left(\int_t^{t+h} e^{-\beta s} c_s\, \mathrm{d}s + V(t+h, x + \int_t^{t+h} (\mu_s - c_s)\, \mathrm{d}s\right)$$

for $\sqrt{\varepsilon} > h > 0$ small enough such that $x + \int_t^{t+h}(\mu_s - c_s)\ \mathrm{d}s \geq 0$. Applying a Taylor expansion we get:

$$-\varepsilon \leq \Big(h e^{-\beta t} c_t + h(V_t(t,x) + (\mu_t - c_t)V_x(t,x)) + o(h)\Big) \leq 0\ .$$

Dividing by h we have

$$-\sqrt{\varepsilon} \leq \Big(e^{-\beta t} c_t + (V_t(t,x) + (\mu_t - c_t)V_x(t,x)) + o(1)\Big) \leq 0\ .$$

Taking $h \to 0$ indicates the following HJB equation for problem (1),

$$\begin{aligned} 0 &= \max\Big(e^{-\beta t} - V_x(t,x), V_t(t,x) + \mu_t\, V_x(t,x)\Big)\ , \qquad (9) \\ 0 &= V(t,x), \quad \text{for } (t,x) \in [0,T]\times(-\infty,0)\ , \\ e^{-\beta T} x &= V(T,x), \quad \text{for } x \geq 0\ . \end{aligned}$$

Since the optimal consumption strategy indicates that there are possible discontinuities of $V(t, x)$ in t and x we may need to show that (9) is fulfilled in a viscosity sense.

Definition 3.1

The upper semi-continuous (usc) envelope of $V(t, x)$ is defined by

$$V^*(t, x) = \limsup_{\substack{(s,y)\to(t,x) \\ (s,y)\in[0,T]\times[0,\infty)}} V(s, y), \quad (t, x) \in [0, T] \times [0, \infty) \ .$$

The lower semi-continuous (lsc) envelope of $V(t, x)$ is defined by

$$V_*(t, x) = \liminf_{\substack{(s,y)\to(t,x) \\ (s,y)\in[0,T]\times[0,\infty)}} V(s, y), \quad (t, x) \in [0, T] \times [0, \infty) \ .$$

Definition 3.2

We say that a linearly bounded function $W : [0, T] \times [0, \infty) \to \mathbb{R}$

- *is a viscosity supersolution if for every $\varphi \in C^{(1,1)}[0, T] \times [0, \infty)$:*

  $$\max\{e^{-\beta\bar{t}} - \varphi_x(\bar{t}, \bar{x}), \varphi_t(\bar{t}, \bar{x}) + \mu_{\bar{t}}\varphi_x(\bar{t}, \bar{x})\} \leq 0,$$

  at every $(\bar{t}, \bar{x}) \in (0, T)\times(0, \infty)$ which is a (strict) minimizer of $W_* - \varphi$ on $[0, T]\times[0, \infty)$ with $W_*(\bar{t}, \bar{x}) = \varphi(\bar{t}, \bar{x})$.

- *is a viscosity subsolution if for every $\psi \in C^{(1,1)}[0, T] \times [0, \infty)$:*

  $$\max\{e^{-\beta\bar{t}} - \psi_x(\bar{t}, \bar{x}), \psi_t(\bar{t}, \bar{x}) + \mu_{\bar{t}}\psi_x(\bar{t}, \bar{x})\} \geq 0 \ ,$$

  at every $(\bar{t}, \bar{x}) \in (0, T)\times(0, \infty)$ which is a (strict) maximizer of $W^* - \psi$ on $[0, T]\times[0, \infty)$ with $W^*(\bar{t}, \bar{x}) = \psi(\bar{t}, \bar{x})$.

W is a viscosity solution if it is both super- and subsolution.

Note: $\psi \geq W^* \geq W \geq W_* \geq \varphi$.

Theorem 3.3

The function $V(t, x)$ given by (1) is a viscosity solution to (9).

For proof see Appendix.

To show the uniqueness of the value function we need the following Lemma, which indicates that some properties of the value function can be transferred to the envelopes.

Lemma 3.4

For $x \in [0, \infty)$ we have

$$V^*(T, x) = V_*(T, x) = e^{-\beta T} x \ .$$

For proof see Appendix.

Next we show the uniqueness of the value function. Since we are dealing with a discontinuous value function, a classical Comparison Theorem common for the continuous case, see for example Bardi and Capuzzo-Dolcetta [4] and references therein, cannot be applied. Therefore, at first we show the uniqueness in the sense of weak comparison, i.e. we show the uniqueness up to discontinuities. The usual technique to prove the uniqueness is to compare the usc envelope u^* of a subsolution u and the lsc envelope v_* of a supersolution v. Because we will be dealing only with continuity regions of the value function it holds $u = u^*$ and $v = v_*$.

Theorem 3.5

Let u be a sub- and v a supersolution to HJB Equation (9), having the properties (P1) – (P6) and fulfilling $u(t, x) \leq v(t, x)$ on $\{0\} \times [0, x] \cup [0, T] \times \{0\}$. Then it holds $u(t, x) \leq v(t, x)$ on R, where $R := [0, T] \times [0, \infty) \backslash S$ with S defined in Remark 2.2.

For proof see Appendix.

Remark 3.6

Theorem 3.5 signifies the uniqueness of the value function in the regions, where it is continuous. Due to Section 2 the value function has only finitely many discontinuities on $[0, T] \times \{0\}$ and finitely many discontinuity curves, which are continuously differentiable functions of time. Furthermore we know that $V(t, x)$ is right continuous in the x component. It is easy to see that the listed properties imply the uniqueness of the value function also on S.

4 Examples

In this section we consider two examples where we calculate the value function explicitly for given drift $\mu(t)$. For the first example we give a detailed construction, by means of the algorithm from Section 2, of the value function. Analogously, but requiring more cumbersome calculations, one can deal with the second one for which we just give the final result and an illustrating plot.

Example 4.1

We choose $T = 3\pi$ and set $\mu_t = \sin(t)$ and $\beta = 0.04$, consequently $M(t) = \int_0^t \sin(s)\ \mathrm{d}s$. The value function V fulfils $V(T, x) = e^{-\beta T} x$ at (T, x).

Using the notation of Section 2 we have that:

$\gamma_0^{(0)} = 0$, $\gamma_1^{(0)} = \infty$, $n = 0$, $\alpha_0^{(0)} = 0$ and $\sin(s) > 0$ on $[3\pi - \varepsilon, 3\pi)$.

Step 1:

Consider

$$\bar{s}^{(1)} := \sup\{s \leq 3\pi : \sin(s) < 0\} = 2\pi \ ;$$

Since $\gamma_1^{(0)} = \infty$, we have $s_1^* = \bar{s}^{(1)} = 2\pi$.

On the set $A_0 := \{(s, x) : 2\pi < s \leq 3\pi, \quad 0 \leq x < \infty\}$ it is optimal to payout the whole surplus immediately. Thus, we can give a closed expression for $V(s, x)$ on the set A_0:

$$\begin{aligned} V(s,x) &= e^{-\beta s} x + \int_s^T e^{-\beta r} \sin(r)\ \mathrm{d}r \\ &= e^{-\beta s} x + \frac{1}{\beta^2 + 1}\Big\{e^{-\beta s}\cos(s) + \beta e^{-\beta s}\sin(s) + e^{-\beta T}\Big\} \ . \end{aligned}$$

Now we are able to calculate the new γ- and α-functions: $\gamma_1^{(1)} = \infty$ and

$$\alpha_0^{(1)}(s) = \frac{1}{\beta^2 + 1}\Big\{e^{-\beta s}\cos(s) + \beta e^{-\beta s}\sin(s) + e^{-\beta T}\Big\}.$$

Step 2:

For $s \in [2\pi - \varepsilon, 2\pi)$ it holds $\sin(s) < 0$ and we set $t = 2\pi$ in the backward algorithm. Like above we calculate $s_2^* = 1.248846988\pi$, $\gamma_1^{(2)}(s) = M(s) - M(2\pi) = 1 - \cos(s)$. Observe that since $\alpha_0^{(1)}(2\pi) > 0$ the point in time s_2^* is bigger than the next change of sign of μ_t, i.e. the discontinuity curve $\gamma_1^{(2)}(s)$ vanishes at this point.

On the set $A_1 := \{(s, x) : s_2^* \leq s < 2\pi, 0 \leq x < 1 - \cos(s)\}$ we have, by the above results, that it is optimal to payout everything immediately, i.e. $V(s, x) = e^{-\beta s} x$, which implies $\alpha_0^{(2)} = 0$.

Step 3:

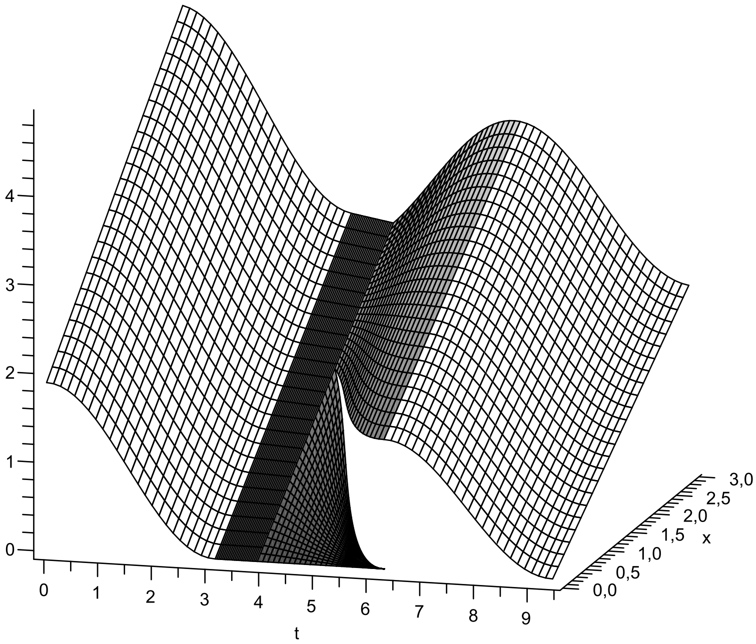

Figure 4: The value function $V(t,x)$ for $\mu_t = \sin(t)$

On the set $\{(s,x) : s_2^* \leq s < 2\pi, x \geq 1-\cos(s)\}$ it is optimal to payout the difference to the $\gamma_1^{(2)}$-curve and to do nothing until $t = 2\pi$.

Thus, we have $V(s, 1-\cos(s)) = \alpha_0^{(1)}(2\pi) = e^{-\beta s}(1-\cos(s)) + \alpha_1^{(2)}$. Now it is easy to calculate $\alpha_1^{(2)}$:

$$\begin{aligned}
\alpha_1^{(2)}(s) : &= \alpha_0^{(1)}(2\pi) - e^{-\beta s}(1-\cos(s)) \\
&= \frac{1}{\beta^2+1}\Big\{e^{-\beta 2\pi}\cos(2\pi) + \beta e^{-\beta 2\pi}\sin(2\pi) + e^{-\beta 3\pi}\Big\} \\
&\quad - e^{-\beta s}(1-\cos(s)) \; .
\end{aligned}$$

Altogether $V(s,x) = e^{-\beta s}x + \alpha_1^{(2)}(s)$ on $\{(s,x) : s_2^* < s \leq 2\pi, x \geq 1-\cos(s)\}$.

Step 4:

It holds $\sin(s) < 0$ in an ε environment of 1.248846988π. We calculate $s_3^* = \pi$, and obtain that $V(t,x) = e^{-\beta t}x$ on $\{(s,x) : \pi \leq s < 1.248846988\pi,\ 0 \leq x < \infty\}$ (in this area one pays out everything and gets ruined!). Therefore $\alpha_0^{(3)} = 0$ and $\gamma_1^{(3)} = \infty$.

Step 5:

For $0 \leq s \leq \pi$ we are in the same the situation like in the beginning of the example, with the consequence that $V(t,x) = e^{-\beta t}x + e^{-\beta s}x + \frac{1}{\beta^2+1}\Big\{e^{-\beta s}\cos(s) + \beta e^{-\beta s}\sin(s) + e^{-\beta\pi}\Big\}$.

Summarizing the results and letting $a := \frac{e^{-\beta 2\pi}+e^{-\beta 3\pi}}{\beta^2+1}$ yields

$$\begin{aligned}
V(t,x) &= e^{-\beta t}x + I_{[0,\pi)}(t)\frac{1}{\beta^2+1}\Big\{e^{-\beta t}\cos(t) + \beta e^{-\beta t}\sin(t) + e^{-\beta\pi}\Big\} \\
&+ I_{[1.2488\pi,2\pi)\times[1-\cos(t),\infty)}(t,x)\Big\{a - e^{-\beta t}(1-\cos(t))\Big\} \\
&+ I_{[2\pi,3\pi)}(t)\frac{1}{\beta^2+1}\Big\{e^{-\beta t}\cos(t) + \beta e^{-\beta t}\sin(t) + e^{-\beta 3\pi}\Big\}\ .
\end{aligned}$$

In Figure 4.1 we see that $V(t,x)$ consists of 5 parts (which have different shadings). Each part corresponds to some dividend payout behaviour of the insurer, which are described in Steps 1 – 5 above. The discontinuity region of $V(t,x)$ is given by

$$D := \Big\{(t,x):\ t \in [1.248846988\pi, 2\pi)\ , x = 1-\cos(t)\Big\}\ .$$

One easily verifies that $V(t,x)$ fulfils (P1) – (P6) and solves the HJB equation (9). ■

Example 4.2

In this example we consider the case where the drift function has a linear component $\mu_t = \sin(t) + 0.01t + 0.2$. Using the same algorithm like in Example 4.1, we obtain a closed form expression for the value function, but calculations in this case are quite tedious.

Let

$$\begin{aligned}
f(t,s) &= \int_t^s (\sin(r) + 0.01r + 0.2)e^{-\beta r}\ \mathrm{d}r, \\
g(t,s) &= \int_t^s (\sin(r) + 0.01r + 0.2)\ \mathrm{d}r.
\end{aligned}$$

Then the value function is given by:

$$\begin{aligned}
&V(t,x) = e^{-\beta t}x + I_{[1.9162\pi,3\pi)}(t)f(t,3\pi) \\
&+ I_{[1.075\pi,1.9162\pi)}(t)I_{[-g(t,1.9162\pi),\infty)}(x)\Big\{f(1.9162\pi,3\pi) + e^{-\beta t}g(t,1.9262\pi)\Big\} \\
&+ I_{[0.524\pi,1.075\pi)}(t)I_{[0,-g(t,1.9162\pi))}(x)f(t,1.075\pi) \\
&+ I_{[0.524\pi,1.075\pi)}(t)I_{[-g(t,1.9162\pi),\infty)}(x)\Big\{f(1.9162\pi,3\pi) + e^{-\beta t}g(t,1.9162\pi)\Big\} \\
&+ I_{[0,0.524\pi]}(t)\Big\{f(t,0.524\pi) + f(1.9162\pi,2\pi))\Big\}.
\end{aligned}$$

The value function now consists of 6 parts, in Figure 4.2 they differ in shadings and correspond to different types of strategies. The upper right, upper left and the both bottom parts correspond to the strategy “payout everything”. The both top centre parts correspond to the strategy “payout the difference to the γ_1 curve and remain on γ_1. Like in the

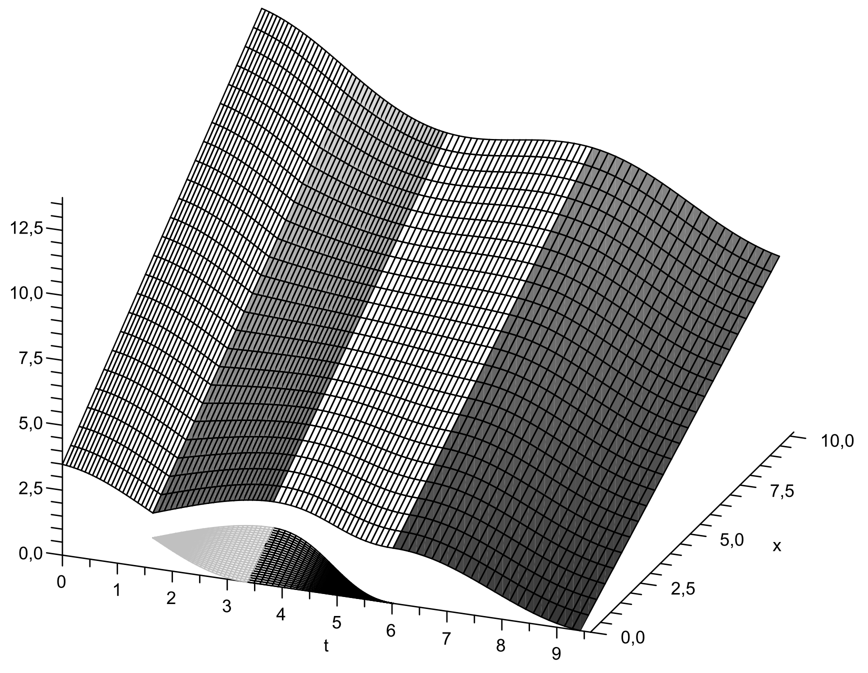

Figure 5: The value function $V(t,x)$ for $\mu_t = \sin(t) + 0.01t + 0.2$.

previous case it is easy to check that $V(t,x)$ fulfils conditions (P1) – (P6) and solves the HJB equation (9). ∎

Appendix

HJB equation - viscosity solution

Proof of Theorem 3.3

We start with the supersolution proof (see the method in Mnif & Sulem [10]).

Let φ be an appropriate test function and $(t,x) \in (0,T)\times(0,\infty)$ such that $V_*(t,x) = \varphi(t,x)$ is a minimizer of $V_* - \varphi$. Let $\{(t_n,x_n)\} \subset (0,T)\times(0,\infty)$ be a sequence with $(t_n,x_n) \to (t,x)$ such that $V(t_n,x_n) \to V_*(t_n,x_n)$ as $n\to\infty$. Since $V \ge V_* \ge \varphi$ we have for a given strategy $C^n \in \mathcal{C}(t_n,x_n)$ some small $h>0$ from (8):

$$\varphi(t_n,x_n) - \varphi(t_n,x_n) + V(t_n,x_n) \ge \int_{t_n-}^{t_n+h\wedge\tau_n-} e^{-\beta s}\,\mathrm{d}C_s^n + \varphi(t_n + h\wedge\tau_n, X^{C^n}_{t_n+h\wedge\tau_n-})\,. \tag{10}$$

We have by the choice of (t_n,x_n) that $\gamma_n = V(t_n,x_n) - \varphi(t_n,x_n) \to V_*(t,x) - \varphi(t,x) = 0$ and $\gamma_n \ge 0$. If we choose $C_s^n = \delta$ for $s \ge t_n$ (one constant payment at time t_n) for $\delta > 0$ such that $X^C_{t_n} = x_n - \delta > 0$. We can choose $\delta > 0$ small enough with $x_n - \delta \ge 0$ for all n.

We obtain by sending $h \to 0$ and $n \to \infty$:

$$\varphi(t,x) \geq e^{-\beta t}\delta + \varphi(t, x-\delta) \ .$$

From which we get $0 \geq e^{-\beta t} - \varphi_x(t,x)$.

If we choose $C_s^n = 0$ for $s \geq t_n$ we obtain from (10)

$$\gamma_n \geq \varphi(t_n + h \wedge \tau, x_n + \int_{t_n}^{t_n+h\wedge\tau} \mu_s \, \mathrm{d}s) - \varphi(t_n, x_n) \ .$$

Since everything is deterministic we can take $h > 0$ small enough such that $x_n + \int_{t_n}^{t_n+h} \mu_s \, \mathrm{d}s \geq 0$ for all n. A Taylor expansion gives

$$\frac{\gamma_n}{h} \geq \varphi_t(t_n, x_n) + \mu_{t_n}\varphi_x(t_n, x_n) + o(1). \tag{11}$$

If $\{\gamma_n\}$ is equal to zero for only finitely many n we take a strictly positive subsequence $\{\gamma_n'\}$ and choose $h = \sqrt{\gamma_n'}$ and n large enough such that there is no ruin before $t_n + h$.
If $\{\gamma_n\}$ is equal to zero for infinitely many n we take a subsequence $\{\gamma_n^*\}$ with $\gamma_n^* = 0$ for $n \in \mathbb{N}$.
We get for (11) if $n \to \infty$

$$0 \geq \varphi_t(t,x) + \mu_t \varphi_x(t,x) \ ,$$

which proves the supersolution property.

For proving the subsolution property we need to show:
For every $\bar{\psi} \in C^{(1,1)}[0,T] \times [0,\infty)$:

$$\max\{e^{-\beta \bar{t}} - \bar{\psi}_x(\bar{t},\bar{x}), \bar{\psi}_t(\bar{t},\bar{x}) + \mu_{\bar{t}}\bar{\psi}_x(\bar{t},\bar{x})\} \geq 0 \ ,$$

at every $(\bar{t},\bar{x}) \in (0,T) \times (0,\infty)$ which is a (strict) maximizer of $V^* - \bar{\psi}$ on $[0,T] \times [0,\infty)$ with $V^*(\bar{t},\bar{x}) = \bar{\psi}(\bar{t},\bar{x})$.
As usual the subsolution proof is done via contradiction. Suppose there are some $(\bar{t},\bar{x})$ and $\bar{\psi}$ with the properties stated before but with

$$\max\{e^{-\beta \bar{t}} - \bar{\psi}_x(\bar{t},\bar{x}), \bar{\psi}_t(\bar{t},\bar{x}) + \mu_{\bar{t}}\bar{\psi}_x(\bar{t},\bar{x})\} < -2\xi \ , \tag{12}$$

for some $\xi > 0$.

Consider the function

$$\psi(t,x) = \bar{\psi}(t,x) + \frac{(x-\bar{x})^2 + (t-\bar{t})^2}{\bar{t}^2 + \bar{x}^2}\xi \ .$$

Then it holds $V^*(\bar{t},\bar{x}) = \bar{\psi}(\bar{t},\bar{x}) = \psi(\bar{t},\bar{x})$, $\bar{\psi}_x(\bar{t},\bar{x}) = \psi_x(\bar{t},\bar{x})$ and $\bar{\psi}_t(\bar{t},\bar{x}) = \psi_t(\bar{t},\bar{x})$ which gives

$$\max\{e^{-\beta\bar{t}} - \psi_x(\bar{t},\bar{x}), \psi_t(\bar{t},\bar{x}) + \mu_{\bar{t}}\psi_x(\bar{t},\bar{x})\} < -2\xi.$$

Because $\psi(t,x)$ is continuously differentiable in both t and x and μ_t is continuous, there is $\delta \in (0, \frac{\sqrt{\bar{t}^2+\bar{x}^2}}{2})$ such that

$$\max\{e^{-\beta t} - \psi_x(t,x), \psi_t(t,x) + \mu_t\psi_x(t,x)\} < -\xi$$

for $(t,x) \in B_\delta(\bar{t},\bar{x})$. We obtain $V^*(t,x) \le \bar{\psi}(t,x) = \psi(t,x) - \frac{\delta^2}{\bar{t}^2+\bar{x}^2}\xi$ for $(t,x) \in \partial B_\delta(\bar{t},\bar{x})$. Let now $\varepsilon = \frac{1}{2}\frac{\delta^2}{\bar{t}^2+\bar{x}^2}\xi$, then on $B_\delta(\bar{t},\bar{x})$ we have

$$\max\{e^{-\beta t} - \psi_x(t,x), \psi_t(t,x) + \mu_t\psi_x(t,x)\} < -\varepsilon\ , \tag{13}$$

while for $(t,x) \notin B_\delta(\bar{t},\bar{x})$ we have

$$V^*(t,x) = \psi(t,x) - 2\,\varepsilon\ .$$

Now let $(t_n,x_n) \to (\bar{t},\bar{x})$ such that $V(t_n,x_n) \to V^*(\bar{t},\bar{x})$ and assume (w.l.g.) that $(t_n,x_n) \in B_\delta(\bar{t},\bar{x})$ for all $n \in \mathbb{N}$.
Let $C^n \in \mathcal{C}(t_n,x_n)$, X^{C^n} be the corresponding wealth starting in (t_n,x_n) and $\tau^* = \tau_n \wedge \bar{T}$ (with some $\bar{T}$ such that $t_n < \bar{T} \le T$ for all n) where

$$\tau_n = \inf\{s \ge t_n \,|\, X_s^{C^n} \notin B_\delta(\bar{t},\bar{x})\}\ .$$

At first we observe X^{C^n} can only have downward jumps in the x direction and that $V(t,x)$ is increasing in x. Because of continuity of μ_t, jumps in the wealth process are due to jumps in the consumption process and we have $X_s^{C^n} - X_{s-}^{C^n} = -\Delta C_s^n$.
Suppose $\tau^* = \tau_n$, i.e. stopping because of leaving $B_\delta(\bar{t},\bar{x})$. Then either we hit the boundary continuously or leave the ball due to a jump at time τ^* in which case $X_{\tau^*-} \in B_\delta(\bar{t},\bar{x})$. From the above estimates we get

$$V(\tau^*, X_{\tau^*-}^{C_n}) \le \psi(\tau^*, X_{\tau^*-}^{C_n}) - 2\,\varepsilon I_{\{X_{\tau^*-}=X_{\tau^*}\}}\ .$$

If $\tau^* = \bar{T}$ then $X_{\tau^*}^{C_n}$ as well $X_{\tau^*-}^{C_n}$ as are still inside the ball and we have

$$V(\tau^*, X_{\tau^*-}^{C_n}) \le \psi(\tau^*, X_{\tau^*-}^{C_n})\ .$$

In total we arrive at

$$
\begin{aligned}
V(\tau^*, X^{C^n}_{\tau^*-}) \leq& \psi(\tau^*, X^{C^n}_{\tau^*-}) - 2\,\varepsilon I_{\{\tau_n=\tau^*\wedge X_{\tau^*-}=X_{\tau^*}\}\}} \\
=& \psi(t_n, x_n) + \int_{t_n}^{\tau^*} \psi_t(s, X^{C^n}_s) + \mu_s \psi_x(s, X^{C^n}_s)\,\mathrm{d}s \\
&- \int_{t_n-}^{\tau^*-} \psi_x(s, X^{C^n}_s)\,\mathrm{d}C^{n,c}_s + \sum_{t_n\leq s<\tau^*,\, X^{C^n}_s \neq X^{C^n}_{s-}} \psi(s, X^{C^n}_s) - \psi(s, X^{C^n}_{s-}) \\
&- 2\,\varepsilon I_{\{\tau_n=\tau^*\wedge X_{\tau^*-}=X_{\tau^*}\}}\,.
\end{aligned}
$$

In the above formula $C^{n,c}$ denotes the continuous part of strategy C^n.

If $X^{C_n}_s \neq X^{C^n}_{s-}$ we have that $X^{C^n}_s - X^{C^n}_{s-} = -\Delta C^n_s$ and we can write

$$
\sum_{t_n\leq s<\tau^*,\, X^{C^n}_s \neq X^{C^n}_{s-}} \psi(s, X^{C^n}_s) - \psi(s, X^{C^n}_{s-}) = - \sum_{t_n\leq s<\tau^*,\, X^{C^n}_s \neq X^{C^n}_{s-}} \left(\int_0^{\Delta C^n_s} \psi_x(s, x-\alpha)\,\mathrm{d}\alpha \right)\,.
$$

Combining the last expression with the continuous part of C^n and using $e^{-\beta t} \leq \psi_x(t,x)$ on $B_\delta(\bar{t}, \bar{x})$ we arrive at

$$
\begin{aligned}
&- \int_{t_n-}^{\tau^*-} \psi_x(s, X^{C^n}_s)\,\mathrm{d}C^{n,c}_s + \sum_{t_n\leq s<\tau^*,\, X^{C^n}_s \neq X^{C^n}_{s-}} \psi(s, X^{C^n}_s) - \psi(s, X^{C^n}_{s-}) \\
&\qquad \leq - \int_{t_n-}^{\tau^*-} e^{-\beta s}\,\mathrm{d}C^{n,c}_s - \sum_{t_n\leq s<\tau^*,\, X^{C^n}_s \neq X^{C^n}_{s-}} \left(e^{-\beta s} \Delta C^n_s \right) = - \int_{t_n-}^{\tau^*-} e^{-\beta s}\,\mathrm{d}C^n_s\,.
\end{aligned}
$$

Finally using $\psi_t(s, X^{C^n}_s) + \mu_s \psi_x(s, X^{C^n}_s) \leq -\varepsilon$ from (13) we get

$$
V(\tau^*, X^{C^n}_{\tau^*-}) + \int_{t_n-}^{\tau^*-} e^{-\beta s}\,\mathrm{d}C^n_s + (\tau^* - t_n)\varepsilon + 2\,\varepsilon I_{\{\tau_n=\tau^*\wedge X_{\tau^*-}=X_{\tau^*}\}} \leq \psi(t_n, x_n)\,,
$$

which is the same as

$$
\begin{aligned}
V(\tau^*, X^{C^n}_{\tau^*-}) + \int_{t_n-}^{\tau^*-} e^{-\beta s}\,\mathrm{d}C^n_s + (\tau^* - t_n)\varepsilon + 2\,\varepsilon I_{\{\tau_n=\tau^*\wedge X_{\tau^*-}=X_{\tau^*}\}} &\leq \psi(t_n, x_n) \\
&\leq V(t_n, x_n) + (\psi(t_n, x_n) - V(t_n, x_n))\,.
\end{aligned}
$$

Since also $\psi(t_n, x_n) \to V^*(\bar{t}, \bar{x})$ if $n \to \infty$ we can choose n large enough such that

$$
0 \leq \psi(t_n, x_n) - V(t_n, x_n) \leq \frac{(\tau^* - t_n)\varepsilon + 2\,\varepsilon I_{\{\tau_n=\tau^*\wedge X_{\tau^*-}=X_{\tau^*}\}}}{2}.
$$

We get

$$
V(\tau^*, X^{C^n}_{\tau^*-}) + \int_{t_n-}^{\tau^*-} e^{-\beta s}\,\mathrm{d}C^n_s + \frac{(\tau^* - t_n)\varepsilon + 2\,\varepsilon I_{\{\tau_n=\tau^*\wedge X_{\tau^*-}=X_{\tau^*}\}}}{2} \leq V(t_n, x_n)\,. \tag{14}
$$

Now before we can state that (14) is a contradiction to (8). We have to discuss the case $\tau^* = \tau_n = t_n$, where an immediate lump-sum consumption leads to $x_n = X^{C^n}_{\tau^*-} > X^{C^n}_{\tau^*} \notin B_\delta(\bar{t}, \bar{x})$. We notice that property (P6) and the construction of the value function show that in this case V is continuously differentiable around the point $(\bar{t}, \bar{x})$ and $V_x(\bar{t}, \bar{x}) = e^{-\beta \bar{t}}$. Therefore $V^* = V$ around $(\bar{t}, \bar{x})$ which furthermore yields that $\psi_x(\bar{t}, \bar{x}) = e^{-\beta \bar{t}}$. Since the test function ψ is continuously differentiable in x, inequality (13) can not be true and states a contradiction to (12).

Thus we have, when stating (12), that there exists an area around $(\bar{t}, \bar{x})$ inside which it is not optimal to consume a lump sum from the wealth. Consequently a strategy C^n, for playing a role in the dynamic programming principle for n large enough such that (t_n, x_n) are inside this non-paying area, has the feature that $X^{C^n}_t$ can leave $B_\delta(\bar{t}, \bar{x})$ through a jump not before leaving the non-paying area continuously.

Therefore $\tau^* > t_n$, which completes the proof and we can conclude that $V(t, x)$ is a viscosity solution to (9). $\square$

Proof of Lemma 3.4

Since $V(t, x) \geq e^{-\beta t} x$ for $(t, x) \in [0, T] \times [0, \infty)$ (you can always payout everything and *quit* by consuming a small constant rate such that $X_{t+} < 0$) we also have

$$V^*(t, x) \geq e^{-\beta t} x \ ,$$
$$V_*(t, x) \geq e^{-\beta t} x \ .$$

From $e^{-\beta T} x = V(T, x) \geq V_*(T, x)$ we get $V_*(T, x) = e^{-\beta T} x$.

Assume that $V^*(T, x) > e^{-\beta T} x$, then there exists some $\eta > 0$ with

$$V^*(T, x) \geq 2\eta + e^{-\beta T} x \ .$$

Now choose a sequence $(t_n, x_n) \to (T, x)$ such that $V(t_n, x_n) \to V^*(T, x)$. There is some $n_0 > 0$ such that for $n \geq n_0$ we have

$$V(t_n, x_n) \geq \eta + e^{-\beta T} x \ . \tag{15}$$

Let $C^n \in \mathcal{C}(t_n, x_n)$ and define $\tau_n = \inf\{t \geq t_n \,|\, X^{C^n}_t < 0\} \wedge T$. Since C^n is admissible we have $\Delta C^n_{t_n} \leq x_n$, there is no lump sum payment leading to ruin. Furthermore by the definition of t_n we have $\tau_n - t_n \to 0$ if $n \to \infty$. Now fix some $\varepsilon > 0$ and choose n large enough such that

$$\int_{t_n-}^{\tau_n-} e^{-\beta r}\,\mathrm{d}C_r^n + e^{-\beta\tau_n} X_{\tau_n-}^{C^n} \le e^{-\beta t_n} C_{t_n}^n + e^{-\beta t_n}(x_n - C_{t_n}^n) + \varepsilon = e^{-\beta t_n} x_n + \varepsilon\ .$$

Taking a supremum over strategies C^n we get $V(t_n, x_n) \le e^{-\beta t_n} x_n + \varepsilon$ which contradicts (15) since ε is arbitrary and if $(t_n, x_n) \to (T, x)$ we have $e^{-\beta t_n} x_n \to e^{-\beta T} x$. $\square$

Proof of Theorem 3.5:

Assume there is $(\hat{t}, \hat{x}) \in R$ such that $u(\hat{t}, \hat{x}) - v(\hat{t}, \hat{x}) > 0$. W.l.o.g. we assume $\hat{t} \in [t_j, t_{j-1})$ and $\hat{x} \in \big(\gamma_{j,i}(\hat{t}), \gamma_{j,i+1}(\hat{t})\big)$ with $t_j < t_{j-1}$, $\gamma_{j,i} < \gamma_{j,i+1}$ defined as in Section 2 and in Remark 2.2. We also assume, that the comparison principle is already shown for the intervals $[t_l, t_{l-1})$ with $l \in \{j-1, ..., m\}$, i.e. $u(t,x) \le v(t,x)$ on $[t_j, T] \times \mathbb{R}_+$. Note that Lemma 3.4 yields $u(x,T) = v(x,T)$ for all $x \in \mathbb{R}_+$.

Define $v^k = kv$ for $k > 1$. It is easy to check, that $k\tilde{v}$ is still a supersolution with lsc envelope kv. Choose $k > 1$ such that $u(\hat{t}, \hat{x}) - v^k(\hat{t}, \hat{x}) > 0$. Due to Lemma 1.1 we obtain the following inequality:

$$\begin{aligned} u(t,x) - v^k(t,x) = u(t,x) - kv(t,x) &\le xe^{-\beta t}(1-k) + \lambda(t) \\ &\le xe^{-\beta t_{j-1}}(1-k) + \lambda(0) =: \eta\ . \end{aligned}$$

It is clear that $u(t,x) - v^k(t,x) \le 0$ for $x \le \frac{\lambda(0)}{k-1} e^{\beta t_{j-1}} =: \eta$. If $\gamma_{j,i+1} = \infty$ on $[t_j, t_{j-1})$ consider

$$A := \{(t,x) : t_j \le t < t_{j-1},\ \ \gamma_{j,i}(t) < x < \eta\}\ .$$

If $\gamma_{j,i+1} < \infty$ on $[t_j, t_{j-1})$ consider

$$A := \{(t,x) : t_j \le t < t_{j-1},\ \ \gamma_{j,i}(t) < x < \gamma_{j,i+1}(t)\}\ .$$

W.l.o.g. we assume $\gamma_{j,i+1}(t) < \infty$ on $[t_j, t_{j-1})$ and $\gamma_{j,l}(t)$, $l \in \{1, ..., n_j\}$, increasing on $[t_j, t_{j-1})$.

Note that due to properties (P5) and (P6) the function $u(t,x) - v^k(t,x)$ is continuously differentiable and decreasing on A. In particular, $\hat{x} \ge \gamma_{j,1}(\hat{t})$.

Define further

$$M := \sup_{(t,x)\in A} \{u(t,x) - v^k(t,x)\}\ .$$

From above we know that $M < \lambda(0) < \infty$ and obtain

$$0 < u(\hat{t}, \hat{x}) - v^k(\hat{t}, \hat{x}) \leq M \ .$$

Since $u - v^k$ is continuous on A there is $(t^*, x^*) \in A$ with $u(t^*, x^*) - v^k(t^*, x^*) > \frac{M}{2} > 0$. Define further $H := \{(t, x, s, y) : (t, x) \ , (s, y) \in A, \ y - x \geq 0 \ , \ t - s \geq 0\}$, $m := \frac{k}{2}$ and for $\xi > 0$:

$$\begin{aligned} f_\xi(t, x, s, y) = {} & u(t, x) e^{\beta s} - v^k(s, y) e^{\beta t} - \frac{\xi}{2}(x - y)^2 \\ & - \Big\{ \frac{2m}{\xi^2(y - x + t - s) + \xi} + \frac{1}{(x - \gamma_{j,i}(t))\xi} + \frac{1}{(\gamma_{j,i+1}(s) - y)\xi} \Big\} \ . \end{aligned}$$

Then it holds

$$f_\xi(t, \gamma_{j,i}(t), s, y) = f_\xi(t, x, s, \gamma_{j,i+1}(s)) = -\infty$$

for $(t, \gamma_{j,i}(t), s, y), (t, x, s, \gamma_{j,i+1}(s)) \in \bar{H}$. Note that $(t, x, s, \gamma_{j,i}(s)), (t, \gamma_{j,i+1}(t), s, y) \in H$ only if $t = s$, which yields $f_\xi(t, x, s, \gamma_{j,i}(s)), f_\xi(t, \gamma_{j,i+1}(t), s, y) < 0$.

Let $M_\xi = \sup_H f_\xi$. Because f_ξ is continuous on H there is $(t_\xi, x_\xi, s_\xi, y_\xi) \in \bar{H}$ such that $M_\xi = f_\xi(t_\xi, x_\xi, s_\xi, y_\xi)$. Since $(t^*, x^*) \in A$, it holds $\gamma_{j,i}(t^*) < x^* < \gamma_{j,i+1}(t^*)$, from which it follows

$$\begin{aligned} M_\xi \geq f_\xi(t^*, x^*, t^*, x^*) \quad & = \quad \big(u(t^*, x^*) - v^k(t^*, x^*)\big) e^{\beta t^*} - \frac{2m}{\xi} \\ & \quad\quad - \frac{1}{(\gamma_{j,i+1}(t^*) - x^*)\xi} - \frac{1}{(x^* - \gamma_{j,i}(t^*))\xi} \\ & > \quad \frac{M}{2} e^{\beta t^*} - \frac{2m}{\xi} - \frac{1}{(\gamma_{j,i+1}(t^*) - x^*)\xi} - \frac{1}{(x^* - \gamma_{j,i}(t^*))\xi} \ . \end{aligned}$$

We obtain directly

$$\begin{aligned} & M_\xi > 0 \ \text{ for } \ \xi > 4 \frac{2m + 1/(x^* - \gamma_{j,i}(t^*)) + 1/(\gamma_{j,i+1}(t^*) - x^*)}{M e^{t^*}} =: \xi_0 \\ & \liminf_{\xi \to \infty} M_\xi \geq \frac{M}{2} > 0 \ . \end{aligned}$$

Next we show that there is ξ_1 such that $(t_\xi, x_\xi, s_\xi, y_\xi) \notin \partial H$ for $\xi \geq \xi_1 \vee \xi_0$.

The boundary of H is given by

$$
\begin{aligned}
\partial H &= \{(t,x,s,y) : (t,x) \in A \,, (s,y) \in \partial A \,, s < t \,, x < y\} \\
&\cup \{(t,x,s,y) : (t,x) \in \partial A \,, (s,y) \in A \,, s < t \,, x < y\} \\
&\cup \{(t,x,s,y) : (t,x),(s,y) \in \bar{A} \,, s \le t \,, x = y\} \\
&\cup \{(t,x,s,y) : (t,x),(s,y) \in \bar{A} \,, s = t \,, x < y\} \,. \qquad (16)
\end{aligned}
$$

Let us first consider the boundary of the set A:

$$
\begin{aligned}
\partial A &= \{(t,\gamma_{j,i+1}(t)) :\ t \in [t_j, t_{j-1}]\} \cup \{(t,\gamma_{j,i}(t)) : t \in [t_j, t_{j-1}]\} \\
&\cup \bigcup_{l=j-1}^{j} \{(t_l, x) :\ x \in [\gamma_{j,i}(t_l), \gamma_{j,i+1}(t_l)]\} \,.
\end{aligned}
$$

Consider the first two sets. By the construction of f_ξ and $\bar{H}$ it holds $f_\xi(t,x,s,y) < 0$ if $x \in \{\gamma_{j,i+1}(t), \gamma_{j,i}(t)\}$ or $y \in \{\gamma_{j,i+1}(s), \gamma_{j,i}(s)\}$.

For $t_l = t_{j-1}$ it holds $f_\xi(t_{j-1},x,s,y), f_\xi(t,x,t_{j-1},y) \le 0$ by the assumption $u(t,x) - v^k(t,x) \le 0$ on $[t_{j-1}, T] \times \mathbb{R}_+$.

It remains to consider $t_l = t_j$. We have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}y} f_\xi(t,x,s,y) &= -k e^{\beta t} e^{-\beta s} - \xi(y-x) + \frac{2m}{\big(\xi(y-x+t-s)+1\big)^2} - \frac{1}{(\gamma_{j,i+1}(s)-y)^2 \xi} \\
&\le -k - \xi(y-x) + 2m - \frac{1}{(\gamma_{j,i+1}(s)-y)^2 \xi} \le 0 \,.
\end{aligned}
$$

That is, $f_\xi(t,x,s,y)$ is decreasing in y. Also it holds

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} f_\xi(t,x,s,x) &= e^{-\beta(t-s)} - k e^{\beta(t-s)} + \frac{1}{(x-\gamma_{j,i}(t))^2 \xi} - \frac{1}{(\gamma_{j,i+1}(s)-x)^2 \xi} \\
&\le 1 - k + \frac{1}{(x-\gamma_{j,i}(t))^2 \xi} \,.
\end{aligned}
$$

Since $f_\xi(t,\gamma_{j,i}(t),s,y) = -\infty$ and f_ξ continuous there is $\delta > 0$ for all $t \in [t_j, t_{j-1}]$ s.t. $f_\xi(t,x,s,y) \le 0$ for $x - \gamma_{j,i}(t) < \delta$. In other words $f_\xi(t,x,s,x)$ is decreasing in x for $x - \gamma_{j,i}(t) \ge \delta$ and $\xi > \frac{1}{\delta^2(k-1)}$. Since $t \ge s$ and $y \ge x$ it holds $(t_j,x,s,y) \in \bar{H} \Rightarrow (t_j,x,s,y) = (t_j,x,t_j,y)$, which gives

$$
\begin{aligned}
f_\xi(t_j,x,s,y) = f_\xi(t_j,x,t_j,y) &\le f_\xi(t_j,x,t_j,x) \\
&\le \begin{cases} f_\xi(t_j,\gamma_{j,i}(t_j),t_j,\gamma_{j,i}(t_j)) < 0 & : \ x - \gamma_{j,i}(t_j) \ge \delta \\ 0 & : \ \text{otherwise} \end{cases} \,.
\end{aligned}
$$

On the other hand because the functions $\gamma_{j,i}, \gamma_{j,i+1}$ are increasing it holds

$$(t, x, t_j, y) \in \bar{H} \Rightarrow \gamma_{j,i}(t) \leq x \leq y \leq \gamma_{j,i+1}(t_j) ,$$

and we can conclude like above $f_\xi(t, x, t_j, y) \leq 0$.

Now we know that $(t_\xi, x_\xi, s_\xi, y_\xi) \in H \backslash \partial H$ for $\xi > \max\{\xi_1, \xi_0\}$, $\xi_1 := \frac{1}{\delta^2(k-1)}$.

Note further that it holds

$$f_\xi(t_\xi, x_\xi, s_\xi, x_\xi) + f_\xi(t_\xi, y_\xi, s_\xi, y_\xi) \leq 2 f_\xi(t_\xi, x_\xi, s_\xi, y_\xi) .$$

Choose now a sequence $\xi_n \to \infty$ such that $(t_{\xi_n}, x_{\xi_n}, s_{\xi_n}, y_{\xi_n}) \to (\bar{t}, \bar{x}, \bar{s}, \bar{y})$. From above we obtain

$$\begin{aligned}
\xi_n\Big((x_{\xi_n} - y_{\xi_n})^2\Big) \leq\ & u(t_{\xi_n}, x_{\xi_n}) e^{\beta s_{\xi_n}} - u(t_{\xi_n}, y_{\xi_n}) e^{\beta s_{\xi_n}} \\
& + v^k(s_{\xi_n}, x_{\xi_n}) e^{\beta t_{\xi_n}} - v^k(s_{\xi_n}, y_{\xi_n}) e^{\beta t_{\xi_n}} \\
& - \frac{y_{\xi_n} - x_{\xi_n}}{(x_{\xi_n} - \gamma_{j,i}(t_{\xi_n}))(y_{\xi_n} - \gamma_{j,i}(t_{\xi_n}))\xi} \\
& - \frac{y_{\xi_n} - x_{\xi_n}}{(\gamma_{j,i+1}(s_{\xi_n}) - x_{\xi_n})(\gamma_{j,i+1}(s_{\xi_n}) - y_{\xi_n})\xi} \\
& + \frac{4m(y_{\xi_n} - x_{\xi_n})}{\big(\xi_n(t_{\xi_n} - s_{\xi_n}) + 1\big)\big(\xi_n(y_{\xi_n} - x_{\xi_n} + t_{\xi_n} - s_{\xi_n}) + 1\big)} \\
\leq\ & - \Big(e^{-\beta(t_{\xi_n} - s_{\xi_n})} + k e^{\beta(t_{\xi_n} - s_{\xi_n})}\Big)(y_{\xi_n} - x_{\xi_n}) \\
& + \frac{4m(y_{\xi_n} - x_{\xi_n})}{\big(\xi_n(t_{\xi_n} - s_{\xi_n}) + 1\big)\big(\xi_n(y_{\xi_n} - x_{\xi_n} + t_{\xi_n} - s_{\xi_n}) + 1\big)} . \qquad (17)
\end{aligned}$$

It is obvious that the right hand side is bounded. Then the left hand side is bounded as well, which is possible only if $(x_{\xi_n} - y_{\xi_n})^2 \to 0$ as $n \to \infty$. We conclude $\bar{x} = \bar{y}$. Taking now the limits on the both sides in (17) yields

$$\lim_{n \to \infty} \xi_n (x_{\xi_n} - y_{\xi_n})^2 \leq 0 ,$$

which implies $\xi_n (x_{\xi_n} - y_{\xi_n})^2 \to 0$. Also we obtain immediately

$$\begin{aligned}
0 \leq & \lim_{\xi_n \to \infty} \xi_n (y_{\xi_n} - x_{\xi_n}) \leq -e^{-\beta(t_{\xi_n} - s_{\xi_n})} - k e^{\beta(t_{\xi_n} - s_{\xi_n})} \\
& + \frac{4m}{\big(\xi_n(t_{\xi_n} - s_{\xi_n}) + 1\big)\big(\xi_n(y_{\xi_n} - x_{\xi_n} + t_{\xi_n} - s_{\xi_n}) + 1\big)} \\
& < 4m .
\end{aligned}$$

Note that $\xi_n(t_{\xi_n} - s_{\xi_n}) \to \infty$ implies $\lim_{\xi_n \to \infty} \xi_n(y_{\xi_n} - x_{\xi_n}) < 0$, which is a contradiction. Thus, we conclude $\lim_{\xi_n \to \infty} \xi_n(t_{\xi_n} - s_{\xi_n}) < \infty$, from which it follows $\bar{t} = \bar{s}$ and $\xi_n(t_{\xi_n} - s_{\xi_n})^2 \to 0$. Note that $\bar{x}$ is bounded away from $\gamma_{j,i+1}(\bar{t})$ and $\gamma_{j,i}(\bar{t})$.

Define the functions

$$\begin{aligned}
\psi(t,x) &= \Big\{\frac{\xi_n}{2}(x - y_{\xi_n})^2 + \frac{2m}{\xi_n^2(y_{\xi_n} - x + t - s_{\xi_n}) + \xi_n} + \frac{1}{(x - \gamma_{j,i}(t))\xi_n}\Big\} \\
&\quad + v^k(s_{\xi_n}, y_{\xi_n})e^{\beta t} + \frac{1}{(y_{\xi_n} - \gamma_{j,i+1}(s_{\xi_n}))\xi_n} + M_{\xi_n} \ , \\
\phi(s,y) &= \ -\Big\{\frac{\xi_n}{2}(x_{\xi_n} - y)^2 + \frac{2m}{\xi_n^2(y - x_{\xi_n} + t_{\xi_n} - s) + \xi_n} + \frac{1}{(x_{\xi_n} - \gamma_{j,i}(t_{\xi_n}))\xi_n}\Big\} \\
&\quad + u(t_{\xi_n}, x_{\xi_n})e^{\beta s} - \frac{1}{(y - \gamma_{j,i+1}(s))\xi_n} - M_{\xi_n} \ .
\end{aligned}$$

These functions are continuously differentiable in t and in x. Furthermore $u(t,x)e^{\beta s_{\xi_n}} - \psi(t,x)$ attains its maximum at (t_{ξ_n}, x_{ξ_n}); $v^k(s,y)e^{\beta t_{\xi_n}} - \phi(s,y)$ attains its minimum at (s_{ξ_n}, y_{ξ_n}). Thus, $\psi(t,x)e^{-\beta s_{\xi_n}}$ and $\phi(s,y)e^{-\beta t_{\xi_n}}$ are test functions for $u(t,x)$ and $v^k(s,y)$ respectively. From

$$\lim_{n\to\infty} \psi_x(t_{\xi_n}, x_{\xi_n}) = \lim_{n\to\infty} \phi_y(s_{\xi_n}, y_{\xi_n}) = 2m = k \ , \tag{18}$$

we conclude that there is $N \in \mathbb{N}$ such that for $n > N$ it holds

$$e^{-\beta t_{\xi_n}} - \psi_x(t_{\xi_n}, x_{\xi_n})e^{-\beta s_{\xi_n}} \le 0 \ , e^{-\beta s_{\xi_n}} - \phi_x(s_{\xi_n}, y_{\xi_n})e^{-\beta t_{\xi_n}} \le 0 \ .$$

Therefore it holds by Definition 3.2 of viscosity sub- and supersolutions:

$$\begin{aligned}
0 &\ \ge\ \phi_s(s_{\xi_n}, y_{\xi_n}) + \mu_{s_{\xi_n}}\phi_y(s_{\xi_n}, y_{\xi_n}) \ , \\
0 &\ \le\ \psi_t(t_{\xi_n}, x_{\xi_n}) + \mu_{t_{\xi_n}}\psi_x(t_{\xi_n}, x_{\xi_n}) \ .
\end{aligned}$$

Subtracting the above inequalities, rearranging the terms and letting $\xi_n \to \infty$ yields the following relation

$$\lim_{n\to\infty} (u(t_{\xi_n}, x_{\xi_n}) - v^k(s_{\xi_n}, y_{\xi_n})) \le 0 \ .$$

On the other hand we know

$$0 < \frac{M}{2} \le \liminf_{\xi\to\infty} M_\xi \le \lim_{n\to\infty} M_{\xi_n} = \lim_{n\to\infty} (u(t_{\xi_n}, x_{\xi_n}) - v^k(s_{\xi_n}, y_{\xi_n})) \ ,$$

which is a contradiction. $\square$

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